Problem Session
Vanderbilt University, May 2000

1. Let P be an n-element poset, with elements labelled $1, 2, 3, ..., n$ so that the sequence $1\,2\,3 \ldots n$ is a linear extension [that is, there is a bijective order-preserving map F from P to the chain $\{1, 2, 3, \ldots, n\}$ such that if $p < q$ in P, then $F(p) < F(q)$]. Such a labelling is called "natural."

   Now consider an arbitrary linear extension of P, that is, a listing of the elements so that, whenever $p < q$ in P, p comes before q in the sequence.

   EXAMPLE. Let P be the 4-element fence $1 < 3 > 2 < 4$. It has 5 linear extensions, 1234, 2134, 1243, 2413, 2143.

   Each linear extension is an element of the symmetric group, S_n. Given a permutation in S_n (a sequence of the numbers 1 through n), a "descent" is a place where a larger number immediately precedes a smaller number. In fact, S_n can be given a lattice ordering as follows: One permutation covers another if the first can be obtained from the second by creating a new descent. The set of all linear extensions of a poset with a natural labelling is a down-set of S_n with this ordering, the "weak Bruhat" order.

   EXAMPLE. The permutations above have 0, 1, 1, 1, and 2 descents, respectively. The lower covers of 2143 are 1243 and 2134.

   Given a naturally-labelled finite poset, let h_k be the number of linear extensions with k descents. (It turns out that h_k depends only on the poset and not on which natural labelling you choose.)

   EXAMPLE. For the 4-element fence, $h_0 = 1$, $h_1 = 3$, $h_2 = 1$.

   CONJECTURE (Edelman, 1981): Is the sequence $h_0, h_1, \ldots$ unimodal? That is, do the numbers increase, and then decrease?

   This conjecture would follow from the Stanley-Neggers Conjecture.

   I believe that the following poset, if it could be constructed, would be a counterexample:

   PROBLEM. Construct a poset with only one linear extension with a lot of descents (say $3n/4$ or more), and many linear extensions with few descents (say $n/4$ or fewer)—in fact, with the property that every permutation below one of them has few descents. (Jonathan Farley)

2. Let S be a finite set and G the undirected graph of subsets of S (in which there is an edge from X to Y if and only if the symmetric difference of X and Y contains a single point). Does there exist a Hamiltonian path in G with the following properties:

   - The path starts at $\emptyset$.
   - When the path arrives at X, at most one subset, Y, of X fails to precede X in the path. In that case, Y immediately follows X in the path.

   (W. T. Trotter)

3. A tournament is a directed graph with a loop at every vertex and with any two vertices having exactly one edge between them. Also, we define a tournament as a groupoid to be a commutative, idempotent groupoid with the property $a \cdot b \in \{a, b\}$. There is an obvious bijection between the two, namely $a \cdot b = a$ iff there is an edge from a to b in the graph. Let the class of all such groupoids be denoted by $\mathcal{T}$. Are all the (finite) subdirectly irreducible tournaments in $\mathsf{HSP}(\mathcal{T})$ actually members of $\mathcal{T}$? (Petar Markovic)

4. A modular ortholattice (MOL) is an ortholattice which is modular as a lattice. In their 1936 paper, 'The Logic of Quantum Mechanics', Birkhoff and von Neumann, suggested the finite height MOLs, essentially the ones coordinatized by finite dimensional vector spaces over (possibly non-commutative) fields, as a possible setting for a non-classical propositional logic of quantum mechanics. Modularity turned out to be too strict a condition for this, but it is interesting to ask whether there are any other interesting examples of MOLs. The type II_1 continuous geometries of Murray and von Neumann are irreducible, uncountable examples of MOLs. But the following question remains open concerning the equational theory of MOLs.

   PROBLEM: Is every variety of MOLs generated by its members of finite height? (Michael Roddy)

5. Is it true that a variety is congruence-join-semidistributive iff it is congruence-meet-semidistributive and satisfies a congruence identity? (Keith Kearnes)

6. If α, β and γ are congruences, define

$$\beta_0 = \beta, \quad \gamma_0 = \gamma,$$
$$\beta_{n+1} = \beta \vee (\alpha \wedge \gamma_n), \text{ and}$$
$$\gamma_{n+1} = \gamma \vee (\alpha \wedge \beta_n).$$

Which locally finite varieties satisfy a congruence identity of the form $\beta_n = \beta_{n+1}$? (Keith Kearnes)

7. Is there an algorithm that accepts as input a finite algebra $\mathbf{A}$ of finite similarity type, and determines whether the quasivariety $ISP(\mathbf{A})$ is finitely axiomatizable? (Ralph McKenzie)